Optimal Coercion in Property Assembly

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Abstract

Multiple units of property, each having an owner with a private value, are perfect complements in a redevelopment project. An optimal mechanism, allowing coercion of owners, is considered for assembly of property. Coercion is manifested in compensations below the reported value of property, but above a predetermined minimum. The mechanism maximizes the joint welfare of owners, and is shown to converge asymptotically, as the number of owners grows, to the first-best at a high rate. The mechanism requires very little knowledge about the distributions of owner valuations, and is robust to misspecification of distributions of both owner or buyer valuations.

Keywords: Property assembly, Coercion, Holdout, Mechanism design

JEL codes: D47, D82, H13

1. Introduction

1.1. Property assembly

Property assembly is a process of purchasing multiple separately owned units of property by a single buyer, to implement a redevelopment project. Each of the units is essential for the buyer: the project cannot be implemented if at least one unit was not purchased. The primary example of property assembly is a purchase of several land tracts for construction of transport infrastructure or large buildings. Kominers and Weyl (2011) provide additional examples, such as patent pools.

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Another important, but rarely discussed, example of property assembly is demolition of aged or out-of-demand condominium buildings, with subsequent redevelopment of land; in this case, the units to be assembled are individual apartments in that building. Unlike land, no building can last forever, so the property assembly process is the most likely end of life of every condominium.

Such assembly of old condominium apartments is especially important in post-Soviet countries where most residential real estate is low-quality multi-unit buildings, "project housing" in the American jargon. Most of them were constructed between 1960s and 1980s and were initially state-owned. In the 1990s, most dwellers were allowed to freely privatize their units. This has created a large and liquid market for individual apartments, but also created the need for property assembly should someone decide to demolish an old building and redevelop the land.

1.2. The Holdout Problem

A well-known difficulty that arises during property assembly is the holdout problem. If assembly is voluntary, an owner of every property unit to be assembled is pivotal, and every such owner will exploit his veto power to extract maximum welfare from the buyer. In doing so, each owner imposes a negative externality on other owners by reducing the chance of a successful transaction. A large body of literature agrees that, as the number of units to be assembled increases, the transaction costs of property assembly increase as well. Some studies have found that the chance of a successful transaction goes to zero as the number of owners increases to infinity. Particularly, Cai (2003), in a model with perfect information but costly negotiations, finds that the negotiation time increases quadratically with the number of owners. Mailath and Postlewaite (1990) solve a Bayesian-Nash game in which multiple owners with private valuations make simultaneous offers;[1] as the number of owners grows large, each offer converges to the upper bound of the value distribution, making the transaction asymptotically impossible. Strange (1995) considers property assembly with heterogenous landowners and concludes that the owners of smaller land tracts will ask more per acre. Cunningham (2013) provides empirical evidence in support of this theory. Miceli and Sirmans (2007) argue that property assembly difficulties cause large-scale development projects to relocate from city centers to periphery, where property is less dispersed; such relocations cause an excessive urban sprawl.

In case of post-Soviet condominiums, which are typically made of dozens or even hundreds of apartments, the above conclusions imply that free-market redevelopment of these buildings is virtually impossible. The prevalence of these buildings in the post-Soviet urban landscape has made impossible the emergence of urban land market, and has dramatically slowed down the evolution of the cities. Free apartment privatization of the 1990s became a blessing for the market of those apartments, but also a curse for the market of land underneath.

1.3. The Existing Solutions to the Holdout Problem

Given the prevalence of the need for property assembly around the world, holdout is truly a trillion-dollar problem. Not surprisingly, scholars and policymakers have long been interested in possible solutions. Many scholars agree that some form of coercion can be applied to unit owners to achieve socially desirable outcomes.

Historically, many governments have been practicing property confiscation with a compensation paid, a practice known as the *eminent domain* (ED henceforth) in the United States. As of today, it remains the only practical method of coercion. In all countries, only the state can initiate such coercion. In traditional market economies such as the United

[1] Mailath and Postlewaite (1990) actually consider multiple buyers purchasing a public good, but their model can be easily modified to a property assembly game, which I do to describe their paper.

States, coercion is limited to public projects such as road construction. Because of that, (i) rights-of-way for roads are rarely assembled by private companies, and (ii) property assembly for private projects, such as demolition of condominium buildings, remains an unsolved problem. In post-Soviet countries, the state does practice property takings for demolition of old condominiums and for subsequent private redevelopment. This practice however remains very limited in scale. For example the city of Moscow demolishes annually less than 100 out of its 40000 multi-unit residential buildings; the decision of which building to demolish is driven not by market forces, but by technical characteristics of the buildings. Other post-Soviet cities demolish only those buildings that are deemed structurally unsafe.

A number of studies have analyzed the economics of ED. Miceli and Segerson (2007) argue that not only the ED itself, but the threat of its use, can facilitate the property assembly process by making owners more cooperative in bargaining. Shavell (2010) concludes, not surprisingly, that ED is more useful when there are more units to be assembled. Calandrillo (2003) proposes to replace the "just compensation" by the state in case of ED taking by the "takings insurance" provided by private insurance companies.

Kominers and Weyl (2011) propose to apply the logic of the Vickrey-Clarke-Groves (VCG) mechanism to property assembly. In their model, there is a fixed set of owners and prospective buyers who simultaneously reveal their valuations of the property to be assembled. The transaction happens if at least one buyer's valuation was above the sum of the owners' valuations. Each party is required to pay the externality it had on all other parties. In particular, that implies that a reluctant owner who has pivotal in preventing the transaction is required to pay a compensation to a buyer and to other owners.

1.4. Criticism of existing approaches

The practice of government takings has well-known deficiencies. Oftentimes, the state decision to take private property is made without the owner's feedback about such decision, which may lead to welfare losses. In many countries, property takings cannot be done for private projects. In other countries, where the state mediates property takings for subsequent private redevelopment, there is little theoretical guidance on how such decisions should be made and how the owners should be compensated. The opportunities for corruption are wide open.

The VCG mechanism of Kominers and Weyl (2011) does condition the assembly transaction on the feedback of the parties involved, and can achieve socially optimal decisions in some ideal conditions. But the actual conditions are rarely ideal; at least three deficiencies of the VCG mechanism should be noted:

- As mentioned earlier, the VCG mechanism requires some owners to pay fees to the buyer(s) if the transaction *did not* happen. Credit constrained owners may be unable to do so.

- Kominers and Weyl (2011) consider property assembly as a one-time event with a fixed set of buyers. At the same time, in many cases (e.g. old condominium demolition) it should be modeled as a continuous event with a flow of buyers: if one buyer has failed to assemble property, another one may appear at some point in the future. The VCG mechanism would then require reluctant owners to pay fees an unlimited number of times, to every prospective buyer.

- As a consequence of the previous point, fraud is likely: fake buyers may emerge whose sole purpose is to collect the VCG compensations from the owners.

1.5. The alternative

As noted above, the common in the literature assumption of a fixed set of prospective buyers (e.g. Cai (2003), Mailath and Postlewaite (1990), Kominers and Weyl (2011)) is not realistic in many cases. A failed property assembly attempt does not mean that the owners will keep their property forever; it only means that they need to wait until another attempt with another buyer. This paper models property assembly as a dynamic game with a constant set of owners, but with a flow of buyers.

As a consequence, a common assumption that owners choose their strategies simultaneously and forever (as in Mailath and Postlewaite (1990)) is not realistic either. In a dynamic environment, people can naturally change their strategy, and restricting them in doing so does not serve any useful purpose. I will assume that owners can update their strategies at any time, and therefore their strategy is based on actually observed, rather than expected, strategies of others. That allows me to replace the complex concept of Bayesian-Nash equilibrium, as in Mailath and Postlewaite (1990), by the simple concept of Nash equilibrium.

Furthermore, to prevent fraud and taxation of credit-constrained owners, I will adopt the "no deal - no cash" concept. No money changes hands unless the property assembly transaction takes place.

In this environment, welfare-improving state coercion is manifested in the fact that the owners' compensation in case of property assembly may be lower than their declared value. Because the buyers' identity is not known before they participate in property assembly, no coercion can be applied to them – otherwise they would not identify themselves as buyers.

Because the coercion is applied only to the owners, it seems unethical to target buyers' welfare as a goal of such coercion. I thus assume that the objective of the authority applying coercion is the aggregate welfare of all owners. The most desired outcome is thus equivalent to the outcome of a bargaining problem in which all owners act as a coalition, and the buyers have zero bargaining power.

2. The model and preliminary results

2.1. Description

2.1.1. The owners

Consider an infinite-horizon continuous time environment. There are n units of property, dispersed between n different owners. During each time period of duration $\mathrm{d}t$, the owner of unit i receives a privately known dollar-valued utility $v_i\mathrm{d}t$ from her property, where the instantaneous value v_i is randomly drawn from a distribution. The private values v_i are independent from each other and are time-invariant. The owners' time preference is given by the discount rate $r > 0$, that is, $e^{-r\mathrm{d}t}$ dollars today are equally preferred to one dollar $\mathrm{d}t$ moments of time later. Owners are risk neutral and maximize the discounted stream of utility.

The analysis below does not require full knowledge of distributions from which v_i were drawn, as obtaining empirical estimates of these distributions might be problematic. In particular, the data from sales of individual units of property is of limited value here, because property assembly typically involves owners who do *not* want to sell their unit at market prices.

Nevertheless, some assumptions about the distributions of owners' values have to be made. We assume that the minimum value v_i^0 for owner i is such that the lowest possible NPV from keeping the object forever, $\frac{v_i^0}{r}$, is equal to the commonly known market value of the object. The justification for such assumption is that owners keep their units (rather than sell them) if and only if their value is above this threshold. Some elements of the model also make use of mean owner valuations, denoted $m_i \equiv Ev_i$. The variances $\sigma_i^2 = E(v_i - m_i)^2$ are also assumed to be positive and finite, although their specific values are of little importance.

Because the holdout problem becomes more pronounced as the number of owners increases, this paper pays special attention to asymptotic properties of the mechanisms analyzed. For that, a proper definition of asymptotics has to be introduced. We will consider infinite sequences of owners, with values randomly drawn from a sequence of distributions. We assume that the distributions are of the same order of magnitude. Specifically, the sequences of above mentioned moments, v_i^0, m_i, σ_i^2 are each bounded from below and from above, and the averages of the mean and variance, $\frac{\sum_{i=1\ldots n} m_i}{n}$ and $\frac{\sum_{i=1\ldots n} \sigma_i^2}{n}$, each converge to a limit denoted m and σ^2, respectively. The lower bound of σ_i^2 is strictly positive.

A statistic of special interest is the average valuation of values, $\bar{v} = \frac{\sum_{i=1\ldots n} v_i}{n}$. Its mean is $\frac{\sum_{i=1\ldots n} m_i}{n}$, converging to m, and its variance is $\frac{\sum_{i=1\ldots n} \sigma_i^2}{n^2}$, converging to zero at the rate of n.

2.1.2. The mechanism

The government requires all owners to publicly announce the value $\tilde{v}_i$ of their unit on a perpetual basis. This value can be updated at any time; thus, we assume that every owner i chooses her $\tilde{v}_i$ knowing the exact values $\tilde{\mathbf{v}}_{-i} \equiv \{\tilde{v}_1, \ldots, \tilde{v}_{i-1}, \tilde{v}_{i+1}, \ldots, \tilde{v}_n\}$ posted by other owners.

Once the values are revealed, the government declares the amount of compensation $x_i(\tilde{\mathbf{v}})$ payable to owner i in case of property assembly, where $\tilde{\mathbf{v}}$ is the vector of all (revealed) valuations. In this paper, I only consider ex-post budget balanced mechanisms, meaning that the buyers' price P is equal to the sum of owners' compensations, for every possible vector of reported values:[2]

$$P(\tilde{\mathbf{v}}) = \sum_i x_i(\tilde{\mathbf{v}}). \tag{1}$$

A mechanism is then a set of owner compensations, as functions of $\tilde{\mathbf{v}}$: $\mathbf{x}(\cdot) = \{x_1(\cdot), \ldots, x_n(\cdot)\}$.

[2]There is a weaker notion of *ex-ante* budget balance, meaning that (1) is true only in expectation. In math, assuming that the values are revealed truthfully $\tilde{\mathbf{v}} = \mathbf{v}$, $E_{\mathbf{v}} P(\mathbf{v}) = E_{\mathbf{v}} \sum_i x_i(\mathbf{v})$. In the latter type of mechanisms, the ex-post budget imbalances $P(\mathbf{v}) - \sum_i x_i(\mathbf{v})$ must be cleared by an insurance agency; the ex-ante budget balance implies its zero expected profit. In practice, such agency may be difficult to create, especially if there is lack of consensus about the distributions of owner valuations.

2.1.3. The buyers

There is an infinite universe of potential buyers. The arrival of buyers is a Poisson process, such that the average time interval between two buyers is unity. Each buyer demands the entire set of n units, any subset has zero value to them. A buyer's value of the set is drawn from a distribution with c.d.f. F_n. We assume that the c.d.f. function is smooth. The distribution is assumed to have a finite mean, for a given n, and to satisfy the following regularity condition:

Assumption 1. *The quantity $z - \frac{1-F_n(z)}{F_n'(z)}$ is strictly increasing for all z such that $F_n(z) < 1$.*

Such assumption is common in the mechanism design literature, e.g. in Myerson and Satterthwaite (1983). The above assumptions also imply that the distribution of values is atomless and has a connected support.

The buyers observe the price defined by (1) and purchase the set of all units if their own value exceeds it. We assume that the buyers do not attempt to bargain, which is quite plausible if the number of owners n is large. Thus, the probability of a successful transaction with a given buyer is $1 - F_n(P)$. If the transaction fails, the owners wait until the next buyer arrives.

No one is forced to make any transfers until the purchase takes place.

The expected welfare of owner i, as a function of the vector of revealed valuations $\tilde{\mathbf{v}}$, can then be described as

$$\begin{aligned} w_i(\mathbf{x}(\cdot), \tilde{\mathbf{v}}, v_i) &= \lim_{\mathrm{d}t \to 0} \mathrm{d}t\,(1 - F_n(P(\tilde{\mathbf{v}})))\,x_i(\tilde{\mathbf{v}}) + (1 - \mathrm{d}t(1 - F_n(P(\tilde{\mathbf{v}}))))\left(v_i \mathrm{d}t + e^{-r\mathrm{d}t} w_i(\mathbf{x}(\cdot), \tilde{\mathbf{v}}, v_i)\right) \\ &= \lim_{\mathrm{d}t \to 0} \frac{\mathrm{d}t\,(1 - F_n(P(\tilde{\mathbf{v}})))\,x_i(\tilde{\mathbf{v}}) + (1 - \mathrm{d}t(1 - F_n(P(\tilde{\mathbf{v}}))))\,v_i \mathrm{d}t}{1 - (1 - \mathrm{d}t(1 - F_n(P(\tilde{\mathbf{v}}))))\,e^{-r\mathrm{d}t}} \\ &= \frac{(1 - F_n(P(\tilde{\mathbf{v}})))x_i(\tilde{\mathbf{v}}) + v_i}{r + 1 - F_n(P(\tilde{\mathbf{v}}))} \end{aligned}$$

It is also useful to define the *gains from trade* as the additional expected welfare relative to that of no-property-assembly scenario:

$$u_i(\mathbf{x}(\cdot), \tilde{\mathbf{v}}, v_i) \equiv w_i(\mathbf{x}(\cdot), \tilde{\mathbf{v}}, v_i) - \frac{v_i}{r} = R_n(P(\tilde{\mathbf{v}})) \left(x_i(\tilde{\mathbf{v}}) - \frac{v_i}{r} \right), \tag{2}$$

where $R_n(P) \equiv \frac{1-F_n(P)}{r+1-F_n(P)}$.

The *aggregate gains from trade* of all owners can be defined as follows:

$$U(P(\cdot), \tilde{\mathbf{v}}, V) \equiv \sum_i u_i(\mathbf{x}(\cdot), \tilde{\mathbf{v}}, v_i) = R_n(P(\tilde{\mathbf{v}})) \left(P(\tilde{\mathbf{v}}) - \frac{V}{r} \right), \tag{3}$$

where $V \equiv \sum_i v_i$ is the sum of owners' valuations.

For asymptotic analysis, we now define how the distribution of buyer valuations compares between settings with different numbers of owners. Denote by $\bar{v} \equiv \frac{V}{n}$ the average property value for its owners. Likewise, denote by $p_n \equiv \frac{P}{n}$ the sale price per owner, in a setting with n owners. We assume that, in two settings with different numbers of owners but identical sale prices per owner, the transaction probability is the same: $F_n(np) = F_1(p), \forall n, p$. As a corollary, the derivatives of F_n are related as follows: $F_n^{(k)}(np) = \frac{1}{n^k} F_1^{(k)}(p)$, where $F_n^{(k)}(\cdot)$ is the k-th derivative of $F_n(\cdot)$. We also have $R_n(np) \equiv \frac{1-F_n(np)}{r+1-F_n(np)} = R_1(p)$, and $R_n^{(k)}(np) = \frac{1}{n^k} R_1^{(k)}(p)$.

2.2. The first best

The government's first best mechanism is the one that maximizes the owners' gains from trade, given true valuations $\mathbf{v}$. If the government could observe $\mathbf{v}$, maximization of (3) would amount to maximization of $R_n(P)\left(P - \frac{V}{r}\right)$ over P, with the following first-order condition of the first-best price P_n^a (assuming an interior solution):

$$R_n'(P_n^a) \left(P_n^a - \frac{V}{r} \right) + R_n(P_n^a) = 0, \tag{4}$$

which defines an implicit function $P_n^a(V)$.

Proposition 1. *Any price P_n^a that satisfies (4) delivers the global maximum of the owners' gains from trade.*

The proof is provided in the Appendix.

We also have that $P_n^a(V)$ is uniquely defined, and that $\frac{\partial P_n^a(V)}{\partial V} > 0$.

How does $P_n^a(\cdot)$ change with n? Define by $p_n^a(v)$ the average first-best price as a function of the average value $\bar{v} \equiv \frac{V}{n}$, $p_n^a(\bar{v}) \equiv \frac{P_n^a(n\bar{v})}{n}$. We can rewrite (4) for a setting with n owners as

$$R_n'(np_n^a(\bar{v}))\left(np_n^a(\bar{v}) - \frac{n\bar{v}}{r}\right) + R_n(np_n^a(\bar{v})) = \frac{1}{n}R_1'(p_1^a(\bar{v}))\left(np_1^a(\bar{v}) - \frac{n\bar{v}}{r}\right) + R_1(p_1^a(\bar{v})) = 0,$$

The above equation uniquely defines p_n^a as a function of $\bar{v}$. Varying n does not affect p_n^a, thus, the optimal price per owner depends on the average valuation $\bar{v}$, but not on the number of owners: $p_n^a(\bar{v}) = p_1^a(\bar{v})$. In other words, the first-best aggregate price P_n^a increases proportionally with n as long as the average owners' valuation $\bar{v}$ remains the same. In the analysis that follows, we drop the subscript n when referring to the average first-best price $p^a(\bar{v})$.

2.3. Unobserved types and asymmetric information

If owner types are their private information, the vector of payments to owners $\mathbf{x}$ must rely on the types reported by owners $\tilde{\mathbf{v}}$. The incentive compatibility then implies that each owner i's utility is maximized with respect to $\tilde{v}_i$, with the following first-order condition:

$$\frac{\partial u_i(\mathbf{x}(\cdot), \tilde{\mathbf{v}}, v_i)}{\partial \tilde{v}_i} = 0,$$

which implicitly defines $\tilde{\mathbf{v}}$ as a function of $\mathbf{v}$. This, in turn, implies that

$$\frac{\mathrm{d}u_i(\mathbf{x}(\cdot), \tilde{\mathbf{v}}, v_i)}{\mathrm{d}v_i} = \frac{\partial u_i}{\partial \tilde{v}_i}\frac{\partial \tilde{v}_i}{\partial v_i} + \frac{\partial u_i}{\partial v_i} = \frac{\partial u_i}{\partial v_i} = -\frac{R_n(P(\tilde{\mathbf{v}}))}{r}, \forall \mathbf{v}. \tag{5}$$

Without loss of generality, we can focus on direct mechanisms in which the vector of reported values $\tilde{\mathbf{v}}$ always equals the vector of true values $\mathbf{v}$. Then, by integrating (5) over v_i, we can present the gains from trade as follows (cf.(2)):

$$R_n(P(\mathbf{v}))\left(x_i(\mathbf{v}) - \frac{v_i}{r}\right) \equiv u_i \equiv u_i^0(\mathbf{v}_{-i}) - \frac{H_i(\mathbf{v})}{r}, \tag{6}$$

where

$$H_i(\mathbf{v}) \equiv \int_{v_i^0}^{v_i} R_n(P(\{z, \mathbf{v}_{-i}\}))\mathrm{d}z \tag{7}$$

and where $u_i^0(\mathbf{v}_{-i})$ is the gains from trade of owner i when his own value is equal to minimum possible, while the values of others are given by $\mathbf{v}_{-i}$.

2.4. Optimal non-coercive mechanism

We now investigate what can be achieved by a mechanism that respects the participation constraints, i.e. allows owners to opt out of the property assembly process. An owner would stay out iff her gains from trade are negative; to keep the owners in, such mechanism would then have to satisfy $u_i(\mathbf{x}(\cdot), \mathbf{v}, v_i) = u_i^0(\mathbf{v}_{-i}) - \frac{H_i(\mathbf{v})}{r} \geq 0, \forall i, \mathbf{v}$. Because $H_i(\mathbf{v})$ increases with v_i from zero to a positive value, the latter constraint is equivalent to

$$u_i^0(\mathbf{v}_{-i}) - \frac{H_i(\{\infty, \mathbf{v}_{-i}\})}{r} \geq 0, \forall i, \mathbf{v}_{-i}. \tag{8}$$

Solving for $u_i^0(\mathbf{v}_{-i})$ in (6), substituting it into (8), and aggregating across all owners, we obtain

$$R_n(P(\mathbf{v}))\left(P(\mathbf{v}) - \frac{V}{1-\beta}\right) + \sum_i \frac{H_i(\mathbf{v}) - H_i(\{\infty, \mathbf{v}_{-i}\})}{r} \geq 0. \tag{9}$$

We now turn to the second-best non-coercive mechanism. It is characterized by maximizing the aggregate gains from trade (3), averaged over the (generally unknown) distribution of owner values, subject to the constraint (9). Note that both (3) and (9) depend on the sum of values V, but not on individual values $\mathbf{v}$, thus we can assume that the second-best aggregate price is a function of aggregate value V only. We denote such price, in a setting with n owners, by $P_n^b(V)$.

Because of that, the function $H_i(\mathbf{v})$ in such mechanism can be redenoted from (7), for a setting with n owners, as $H_i(\mathbf{v}) = H_n^b(V) - H_n^b(\sum_{j\neq i} v_j + v_i^0)$, where

$$H_n^b(V) \equiv \int_0^V R_n(P_n^b(V))\mathrm{d}V. \tag{10}$$

With new notations, (9) divided by n becomes

$$R_n(np_n^b(\bar{v}))\left(p_n^b(\bar{v}) - \frac{\bar{v}}{r}\right) + \frac{H_n^b(n\bar{v}) - H_n^b(\infty)}{r} \geq 0, \tag{11}$$

where $p_n^b(\bar{v}) \equiv \frac{P_n^b(n\bar{v})}{n}$.

We now repeat the impossibility result of Mailath and Postlewaite (1990) in the setting of this paper.

Proposition 2. *For all but a countable number of realizations of $\bar{v}$, the probability of transaction converges to zero with n.*

Proof. Suppose the contrary, that there exists a subset $\mathcal{V} \subseteq R_+$ satisfying $|\mathcal{V}| \equiv \int_{v\in\mathcal{V}} \mathrm{d}v > 0$ and such that, for every $\bar{v} \in \mathcal{V}$ there exists an infinite increasing sequence $n_1, \ldots, n_k, \ldots$ with bounded away from zero probability of transaction. Then, there exists $\epsilon > 0$ such that $R_{n_k}(n_k p_{n_k}^b(\bar{v})) > \epsilon, \forall k, \forall \bar{v} \in \mathcal{V}$. By integrating the latter inequality over all $\bar{v} \in \mathcal{V}$ and recalling (10), we obtain $H_{n_k}^b(\infty) - H_{n_k}^b(n_k v_0) \geq \int_{v\in\mathcal{V}} R_{n_k}(n_k p_{n_k}^b(v))\mathrm{d}n_k v \geq n_k \epsilon |\mathcal{V}|$, where $v_0 = \inf \mathcal{V}$. Thus, the component $\frac{H_{n_k}^b(n_k v_0) - H_{n_k}^b(\infty)}{r}$ in (11) is smaller than $-n_k \frac{\epsilon}{r}|\mathcal{V}|$ which converges to negative infinity with k.

At the same time, the first component of (11) is

$$R_{n_k}(n_k p_{n_k}^b(\bar{v}))\left(p_{n_k}^b(\bar{v}) - \frac{\bar{v}}{r}\right) = R_1(p_{n_k}^b(\bar{v}))\left(p_{n_k}^b(\bar{v}) - \frac{\bar{v}}{r}\right) < \frac{1 - F_1(p_{n_k}^b(\bar{v}))}{r} p_{n_k}^b(\bar{v}). \tag{12}$$

Because the buyer's value is assumed to have a mean, $1 - F_1(x)$ must approach zero at a rate faster than n, meaning that $(1 - F_1(x))x$ decreases for large enough x, and therefore has a finite upper bound. Thus, the first (positive) component of (11) is bounded from above, while the second goes to negative infinity, violating the inequality for all sufficiently large n_k. ■

3. Coercive mechanisms: preliminaries

3.1. Description

The above discussion implies that coercion is necessary for optimality. Coercion is the removal of participation constraint (8), resulting in negative gains from trade for some owners. While many experts agree that *some* coercion is necessary in property assembly, there is also a consensus that *too much* coercion is not acceptable. For example, the Fifth Amendment to the U.S. constitution, the most famous piece of legislation regarding property takings, states "[No] private property [shall] be taken for public use, without just compensation." In legal practice, the "just compensation" is usually the market value of the confiscated unit. Compensations paid in other countries are usually calculated in the same way.

In the current model, the assumed market value of unit i is $\frac{v_i^0}{r}$. To make a coercive mechanism consistent with existing legislation, it is then necessary that

$$x_i(\mathbf{v}) \geq \frac{v_i^0}{r}, \forall i, \forall \mathbf{v}. \tag{13}$$

Throughout the paper, we refer to (13) as the *constitutional constraint*, which is a weaker substitute of the participation constraint (8).

I also include some notion of equity between owners. To make sure that a large proportion of gains from trade is not regularly channeled to one or few owners, I assume there exists an upper bound $\bar{x}$ on the expected payment to each owner i, given realizations of the values of others:

$$E_{v_i} x_i(\mathbf{v}) \leq \bar{x}. \tag{14}$$

This assumption is justified by the fact that the expected value m_i of owner i is bounded from above. Also note that, even if this restriction was not imposed, only a vanishing fraction of owners could afford an unbounded compensation per owner, because the expected aggregate gains from trade are always of the same order of magnitude as the expected aggregate value.

3.2. Fixed compensation mechanism

It is easy to find a coercive mechanism that meets the above specified constraints and delivers relatively high levels of welfare. For example, consider a *fixed compensation* mechanism which completely ignores owners' inputs and offers them compensations $\{x_1, \ldots, x_n\}$ that respect (13) and (14). The owners' joint gains from trade are then $R(P)\left(P - \frac{V}{1-\beta}\right)$, where $P = \sum_i x_i$. Maximizing the expected joint gains from trade yields the optimal per-owner price p^c such that (cf.(4))

$$R_n'(np_n^c)\left(np_n^c - \frac{n\bar{m}_n}{r}\right) + R_n(np_n^c) = 0, \tag{15}$$

where $\bar{m}_n \equiv \frac{\sum_{i=1}^n m_i}{n}$ is the expectation of $\bar{v}$ over owners' valuations. By assumptions of the paper, it converges to m.

How good is this mechanism? Define the *loss* function as the difference between the first-best gains from trade per owner and the one delivered by the mechanism:

$$L_n^c(\bar{v}) \equiv R_n(np^a(\bar{v}))\left(p^a(\bar{v}) - \frac{\bar{v}}{r}\right) - R_n(np_n^c)\left(p_n^c - \frac{\bar{v}}{r}\right).$$

The loss function satisfies $L_n^c(\bar{m}_n) = 0$, $L_n^c(\bar{v}) \geq 0$, $L_n^{c\prime}(\bar{v}) = -\frac{R_n(np^a(\bar{v})) - R_n(np_n^c)}{r}$ increasing in v, and $L_n^{c\prime}(\bar{m}_n) = 0$.

Although the asymptotic loss for a given $\bar{v}$ is generally non-zero, the *ex-ante* loss (i.e. its expectation over $\bar{v}$) is, because the difference $\bar{v} - \bar{m}_n$ converges in probability to zero by the law of large numbers. In other words, even a mechanism as simple as fixed compensation is asymptotically efficient, provided that the aggregate compensation was chosen correctly.

We now analyze how fast the expected loss converges to zero. We approximate $L_n^c(\bar{v})$ by the second-order Taylor expansion around $\bar{m}_n$: $L_n^c(\bar{v}) = \frac{1}{2}L^{c\prime\prime}(\bar{m}_n)(\bar{v} - \bar{m}_n)^2 + o((\bar{v} - \bar{m}_n)^2)$. Then,

$$E_{\bar{v}}L^c(\bar{v}) \approx \frac{1}{2}L^{c\prime\prime}(\bar{m}_n)E_{\bar{v}}\int(\bar{v} - \bar{m}_n)^2 = \frac{1}{2}L^{c\prime\prime}(\bar{m}_n)\frac{\sum_{i=1}^n \sigma_i^2}{n^2},$$

which converges to zero at the rate of n. In simple words, as the number of owners doubles, the magnitude of the ex-ante aggregate loss remains unchanged, so that the ex-ante loss per owner is cut in half.

3.3. Own-compensation mechanism

We now consider a mechanism in which the compensation x_i to owner i may depend on her own reported type $\tilde{v}_i$, as well as on compensations to other owners $x_j, j \neq i$, which i takes as given, but not directly on the types of other owners. Then, an owner can manipulate his own compensation, but cannot directly manipulate that of others.

Because an owner i can influence the probability of transaction only through own compensation, the incentive compatibility constraint is as follows:

$$\frac{\partial u_i(\mathbf{x}(\cdot), \tilde{\mathbf{v}}, v_i)}{\partial \tilde{v}_i} = \left(R_n'(P)\left(x_i - \frac{v_i}{r}\right) + R_n(P)\right)\frac{\partial x_i}{\partial \tilde{v}_i} = 0.$$

Focusing on direct mechanisms where $\tilde{\mathbf{v}} = \mathbf{v}$, the latter equality can be true only if at least one of the following is true:

$$R_n'(P)\left(x_i - \frac{v_i}{r}\right) + R_n(P) = 0, \tag{16}$$

meaning that owner's i utility (conditional on choices of others) is maximized, or $\frac{\partial x_i}{\partial v_i} = 0$, meaning that the compensation is (locally) constant.

Denote by $x_i^*(v_i, \mathbf{x}_{-i})$ the compensation determined by (16). It is strictly increasing in v_i. Owner i wants his compensation x_i to be as close as possible to $x_i^*(\cdot)$. Other owners want x_i to be as low as possible if their own gains from trade are positive (to increase the transaction probability through lower aggregate price), and as high as possible if their own gains from trade are negative.

Proposition 3. *For each owner i and all realizations of value $\mathbf{v}$, the optimal compensation $x_i(v_i, \mathbf{x}_{-i})$ never exceeds $x_i^*(v_i, \mathbf{x}_{-i})$.*

In other words, coercion can be manifested in lowering the compensation paid to an owner, but never in increasing it. The proof is in the Appendix.

Therefore, if a compensation to owner i is not equal to $x_i^*(v_i, \mathbf{x}_{-i})$, it must be lower than that and not dependent on v_i. Denote such compensation $\hat{x}_i(\mathbf{x}_{-i})$; this is the maximum that owner i can get given $\mathbf{x}_{-i}$.

Such compensation scheme can presented in a very simple and intuitive way: every owner i is offered a maximum compensation $\hat{x}_i(\mathbf{x}_{-i})$ in case of property assembly transaction, but is allowed to ask less than that in order to increase the chance of transaction. Owners with sufficiently low value v_i, such that $x_i^*(v_i, \mathbf{x}_{-i}) < \hat{x}_i(\mathbf{x}_{-i})$ will then take advantage of the opportunity.

Proposition 4. *For any sequence of mechanisms with increasing number of owners n, with per-owner compensations p_n converging to p such that $R_1(p) > 0$, we have that the compensation to each owner i equals $\hat{x}_i(\mathbf{x}_{-i})$ when n is large enough.*

In other words, as the number of owners increases, to achieve a non-vanishing probability of transaction, all owners earn a fixed compensation that does not depend on their (reported) value. Thus, an own-compensation mechanism with non-vanishing gains from trade is identical to a fixed-compensation mechanism when the number of owners is large enough.

Proof. Suppose the contrary, that for each element in an infinite sequence $n_1, \ldots, n_k, \ldots$ there exists an owner i whose compensation is not equal to $\hat{x}_i(\mathbf{x}_{-i})$, meaning that it must be equal to $x_i^*(v_i, \mathbf{x}_{-i})$ as defined by (16). By assumption, $R_{n_k}(n_k p_{n_k})$ is not zero for large k, and so is its derivative, so we can rewrite (16) for an owner i with the lowest possible value v_i^0 as follows:

$$x_i^*(v_i^0, \mathbf{x}_{-i}) = \frac{R_{n_k}(n_k p_{n_k})}{-R'_{n_k}(n_k p_{n_k})} + \frac{v_i^0}{r} = n_k \frac{R_1(p_{n_k})}{-R'_1(p_{n_k})} + \frac{v_i^0}{r} \to_{k\to\infty} \infty.$$

Therefore, (14) is violated for all sufficiently large k. Because the compensation to owner i is non-decreasing in v_i, (14) is also violated for all v_i, for all sufficiently large k. ■

4. Optimal mechanism

We now consider mechanisms which make the compensation x_i depend on the entire vector of reported values $\tilde{\mathbf{v}}$. The goal of this section is to find out the best such mechanism that respects all constraints and does not require the knowledge of how the owners' valuations are distributed.

What exactly prevents us from achieving the first-best price $P_n^a(V)$ determined by (4)? Because owner values are private information, the incentive compatibility constraint (6) must be respected. This leads us to two issues, each of which prevents the first-best from being achieved.

The first issue is that of division of gains from trade between owners. Suppose the mechanism is such that the buyer's price is first-best, $P_n^a(V)$. Then, the function $H_i(\mathbf{v})$ in (6) can be redefined as follows:

$$H_i(\mathbf{v}) = H_n^a(V) - H_n^a(v_i^0 + \sum_{j\neq i} v_j),$$

where $H_n^a(V) \equiv \int_0^V R_n(P_n^a(V))$. With new notations, (6) becomes

$$R_n(P_n^a(V)) \left(x_i(\mathbf{v}) - \frac{v_i}{r} \right) \equiv u_i^0(\mathbf{v}_{-i}) - \frac{H_n^a(V) - H_n^a(V + v_i^0 - v_i)}{r}. \tag{17}$$

Aggregate (17) across all owners and rearrange to obtain

$$\sum_i u_i^0(\mathbf{v}_{-i}) \equiv R_n(P_n^a(V))\left(P_n^a(V) - \frac{V}{r}\right) + n\frac{H_n^a(V)}{r} - \sum_i \frac{H_n^a(V + v_i^0 - v_i)}{r}. \tag{18}$$

Take a cross-derivative of both sides of (18) with respect to $v_1, \ldots, v_n$. The left-hand side is then necessarily zero, because each element of the left-hand side depends on all $v_1, \ldots, v_n$ except one. At the same time, the right-hand side is generally non-zero, making the first-best allocation for all realizations of $\mathbf{v}$ generally impossible.

The second issue is related to the constitutional constraint (13). Section 2.4 has determined that coercion is necessary for asymptotically positive gains from trade, and therefore at least some owners will have negative gains from trade (17) for some realizations of $\mathbf{v}$. For any such owner i, because the right-hand side of (17) is non-increasing in own value v_i, it is bounded away from zero (downwards) as v_i goes to its upper bound. At the same time, the left-hand side of (17) satisfies

$$\begin{aligned} R_n(P_n^a(V))\left(x_i(\mathbf{v}) - \frac{v_i}{r}\right) &\underbrace{\geq}_{(13)} -R_n(P_n^a(V))\frac{v_i - v_i^0}{r} = -\frac{1 - F_n(P_n^a(V))}{r + 1 - F_n(P_n^a(V))}\frac{v_i - v_i^0}{r} \\ &\geq -\frac{(1 - F_n(P_n^a(V)))(v_i - v_i^0)}{r^2} \geq_{P_n^a(V) \geq V} -\frac{(1 - F_n(V))(v_i - v_i^0)}{r^2} \to_{v_i \to \infty} 0. \end{aligned} \tag{19}$$

The latter limit is true because $1 - F_n(V) = o(v_i^{-1})$; otherwise, the distribution of buyers' values would not have a mean, contradicting the assumption of Section 2.1.3.

Therefore, for large enough v_i, the left and right-hand sides of (17) disagree under the first-best mechanism due to the constitutional constraint (13).

The analysis that follows offers a remedy to each of these two problems and provides an estimate of the cost of each remedy, due to a deviation from the first-best.

4.1. Dividing the gains from trade

Throughout this section, we assume that the constitutional constraint (13) is not binding for all owners, i.e. each of the owner values is below a certain threshold. The other scenario is discussed in section 4.2.

Denote the right-hand side of (18) by $G_n(\mathbf{v})$; our goal is to approximate it by a sum of n functions, such that function i depends on all elements of $\mathbf{v}$ except v_i. Different orders $k = 1 \ldots n$ of approximation are available. For each setting with n owners, define the *weight* of owner i as the mean value of that owner, relative to the sum of means: $s_{i,n} \equiv \frac{m_i}{\sum_j m_j}$. As the number of owners increases, the weight of each owner decreases to zero at the rate of n.

4.1.1. The approximation: definition

The first-order approximation of $u_i^0(\mathbf{v}_{-i})$ can then be introduced as follows (dropping the subscript n):

$$\tilde{u}_i^{[1]}(\mathbf{v}) = u_i^{[1]}(\mathbf{v}_{-i}) + \eta_i^{[1]}(\mathbf{v}),$$

where $u_i^{[1]}(\mathbf{v}_{-i}) \equiv s_i G(\{m_i, \mathbf{v}_{-i}\})$ and $\eta_i^{[1]}(\mathbf{v}) \equiv s_i \epsilon_i^{[1]}(\mathbf{v})$ such that $\epsilon_i^{[1]}(\mathbf{v}) \equiv G(\mathbf{v}) - G(\{m_i, \mathbf{v}_{-i}\})$. It is trivial to verify that $\sum_i \tilde{u}_i^{[1]}(\mathbf{v}) = G(\mathbf{v})$.

This approximation is not perfect because u_i^0 is meant to be independent of v_i, while $\tilde{u}_i^{[1]}$ indeed depends on it via $\epsilon_i^{[1]}$. A better second-order approximation can be achieved by approximating the latter for each owner i by a sum of $n-1$ functions,

$$\epsilon_i^{[1]}(\mathbf{v}) = \sum_{i_2 \neq i} \frac{s_{i_2}}{1 - s_i} \left[\epsilon_i^{[1]}(\{m_{i_2}, \mathbf{v}_{-i_2}\}) + \epsilon_{i,i_2}^{[2]}(\mathbf{v}) \right], \tag{20}$$

and by attributing each of these functions to an owner other than i. The second-order approximation of $u_i^0(\mathbf{v}_{-i})$ is then $\tilde{u}_i^{[2]}(\mathbf{v}) = u_i^{[2]}(\mathbf{v}_{-i}) + \eta_i^{[2]}(\mathbf{v})$ such that

$$u_i^{[2]}(\mathbf{v}_{-i}) \equiv u_i^{[1]}(\mathbf{v}_{-i}) + s_i \sum_{i_2 \neq i} \frac{s_{i_2}}{1 - s_{i_2}} \epsilon_{i_2}^{[1]}(\{m_i, \mathbf{v}_{-i}\})$$

and $\eta_i^{[2]}(\mathbf{v}) \equiv s_i \sum_{i_2 \neq i} \frac{s_{i_2}}{1-s_i} \epsilon_{i,i_2}^{[2]}(\mathbf{v})$. The latter is defined by (cf.(20)) $\epsilon_{i_1,i_2}^{[2]}(\mathbf{v}) \equiv \epsilon_{i_1}^{[1]}(\mathbf{v}) - \epsilon_{i_1}^{[1]}(\{m_{i_2}, \mathbf{v}_{-i_2}\})$. It is, again, straightforward to verify that $\sum_i \tilde{u}_i^{[2]}(\mathbf{v}) = G(\mathbf{v})$.

By induction, the k-th order approximation of $u_i^0(\mathbf{v}_{-i})$ is

$$\tilde{u}_i^{[k]}(\mathbf{v}) = u_i^{[k]}(\mathbf{v}_{-i}) + \eta_i^{[k]}(\mathbf{v}) \tag{21}$$

such that

$$u_i^{[k]}(\mathbf{v}_{-i}) \equiv u_i^{[k-1]}(\mathbf{v}_{-i}) + s_i \sum_{i_2 \neq i} \frac{s_{i_2}}{1-s_{i_2}} \cdots \sum_{i_k \neq i, i_2, \ldots, i_{k-1}} \frac{s_{i_k}}{1 - \sum_{l=2\ldots k} s_{i_l}} \epsilon_{i_2,\ldots,i_k}^{[k-1]}(\{m_i, \mathbf{v}_{-i}\})$$

and

$$\eta_i^{[k]}(\mathbf{v}) = s_i \sum_{i_2 \neq i} \frac{s_{i_2}}{1-s_i} \cdots \sum_{i_k \neq i, i_2, \ldots, i_{k-1}} \frac{s_{i_k}}{1 - s_i - \sum_{l=2\ldots k-1} s_{i_l}} \epsilon_{i,i_2,\ldots,i_k}^{[k]}(\mathbf{v}). \tag{22}$$

The k-th order error is given by $\epsilon_{i_1,\ldots,i_k}^{[k]}(\mathbf{v}) = \epsilon_{i_1,\ldots,i_{k-1}}^{[k-1]}(\mathbf{v}) - \epsilon_{i_1,\ldots,i_{k-1}}^{[k-1]}(\{m_{i_k}, \mathbf{v}_{-i_k}\})$. Because in the above approximation sequence each $i_1, \ldots, i_k$ must be unique, the highest-possible order of approximation is n.

4.1.2. The approximation: asymptotic properties

We now assess the asymptotic properties, as the number of owners n goes to infinity, of the approximation error $\eta_i^{[k]}$ of arbitrary order k. First of all, we study the function $G_n(\mathbf{v})$ and its components. Observe that

$$H_n^a(n\bar{v}) \equiv n H_1^a(\bar{v}), \forall \bar{v}. \tag{23}$$

For proof, first observe that $H_n^a(0) = 0, \forall n$. Next, differentiate both sides of (23) w.r.t. $\bar{v}$ to observe identity of the derivatives:

$$\frac{\partial H_n^a(n\bar{v})}{\partial \bar{v}} = n \frac{\partial H_n^a(n\bar{v})}{\partial n\bar{v}} = n R_n(n p^a(\bar{v})) = n R_1(p^a(\bar{v})) = n \frac{\partial H_1^a(\bar{v})}{\partial \bar{v}}, \forall \bar{v}.$$

By Taylor-expanding (23), we have

$$\begin{aligned} H_n^a(V+v_i^0-v_i) &= nH_1^a(\bar{v}-\frac{1}{n}(v_i-v_i^0)) \\ &= nH_1^a(\bar{v}) - nR_1(p^a(\bar{v}))\frac{1}{n}(v_i-v_i^0) + n\sum_{l=2\ldots\infty}\frac{1}{l!}R_1^{(l-1)}(p^a(\bar{v}))\left(-\frac{v_i-v_i^0}{n}\right)^l. \end{aligned} \tag{24}$$

Using this information, we can rewrite $G_n(\mathbf{v})$ as follows:

$$G_n(\mathbf{v}) = nR_1(p^a(\bar{v}))\left(p^a(\bar{v}) - \frac{\sum_i v_i^0}{nr}\right) - \frac{n}{r}\sum_{l=2\ldots\infty}\frac{1}{l!}R_1^{(l-1)}(p^a(\bar{v}))\sum_{i=1}^{n}\left(-\frac{v_i-v_i^0}{n}\right)^l. \tag{25}$$

The first component on the right-hand side of (25) is asymptotically proportional to n, while other components are of smaller and ever-decreasing order of magnitude. This allows us to approximate $G_n(\mathbf{v})$ by the first component of (25),

$$G_n(\mathbf{v}) \approx ng_n(\bar{v}), \tag{26}$$

where $g_n(z) \equiv R_1(p^a(z))\left(p^a(z) - \frac{\sum_i v_i^0}{nr}\right)$.

Next, we calculate the asymptotic order of magnitude of $\epsilon_i^{[1]}$. From (26),

$$G_n(\{m_i, \mathbf{v}_{-i}\}) \approx ng_n(\bar{v} - \frac{1}{n}(v_i - m_i)) = ng_n(\bar{v}) - g_n'(\bar{v})(v_i - m_i) + o(v_i - m_i),$$

thus $\epsilon_i^{[1]}$ can be approximated by

$$\epsilon_i^{[1]}(\mathbf{v}) \approx g_n'(\bar{v})(v_i - m_i),$$

with asymptotic rate of n^0, i.e. converging to a constant value.

By induction, we conjecture that $\epsilon^{[k]}_{i_1,\dots,i_k}(\mathbf{v})$ can be approximated by

$$\epsilon^{[k]}_{i_1,\dots,i_k}(\mathbf{v}) \approx n^{-k+1} g_n^{(k)}(\bar{v}) \prod_{l=1\dots k} (v_{i_l} - m_{i_l}), \tag{27}$$

where $g_n^{(k)}(\cdot)$ is the k-th derivative of $g_n(\cdot)$. Suppose this is true for $\epsilon^{[k-1]}_{i_1,\dots,i_{k-1}}(\mathbf{v})$. Then,

$$\begin{aligned}
\epsilon^{[k-1]}_{i_1,\dots,i_{k-1}}(\{m_{i_k}, \mathbf{v}_{-i_k}\}) &= n^{-k+2} \prod_{l=1\dots k-1} (v_{i_l} - m_{i_l}) \left[g_n^{(k-1)}(\bar{v} - \frac{1}{n}(v_{i_k} - m_{i_k})) \right] \\
&= n^{-k+2} \prod_{l=1\dots k-1} (v_{i_l} - m_{i_l}) \left[g_n^{(k-1)}(\bar{v}) - \frac{1}{n} g^{(k)}(v_{i_k} - m_{i_k}) + o\left(\frac{v_{i_k} - m_{i_k}}{n} \right) \right].
\end{aligned}$$

By referring to the definition of $\epsilon^{[k]}_{i_1,\dots,i_k}(\mathbf{v})$, we can approximate it by (27).

Next, observe from (22) that every $\eta_i^{[k]}(\mathbf{v})$ is a weighted average of $\epsilon^{[k]}_{i,i_2,\dots,i_k}(\mathbf{v})$, for various sequences of $i_2 \dots i_k$, multiplied by the weight $s_{i,n}$. Because each element $\epsilon^{[k]}_{i,i_2,\dots,i_k}(\mathbf{v})$ of the weighted sum has order of magnitude $n^{-(k-1)}$ while the weight $s_{i,n}$ has order of magnitude n^{-1}, we conclude that $\eta_i^{[k]}(\mathbf{v}) = O(n^{-k})$.

4.1.3. Inefficiency due to approximation error

Because each function $u_i^0(\mathbf{v}_{-i})$, independent from v_i, is approximated by $\tilde{u}_i^{[k]}(\mathbf{v})$, which does depend on v_i via $\eta_i^{[k]}(\mathbf{v})$, owners will deviate from reporting their true value v_i.[3] This section characterizes the asymptotic properties of the reported value deviation and of the associated deviation of welfare from the first-best.

The compensation to owner i as a function of the vector of reported types $\tilde{\mathbf{v}}$ is determined from (17), assuming that the types are reported truthfully, the aggregate price is first-best,

[3] An equivalent direct mechanism can be specified in which owners report their true value. Such mechanism however is much more computationally intensive, because its aggregate price P is a function of n-dimensional vector of reported valuations. In contrast, in the optimal "almost direct" mechanism of this paper, the aggregate price is $P_n^a(\tilde{V})$, i.e. depends only on the scalar sum of reported valuations, and is straightforward to calculate from (4).

transaction probability is positive, and $u_i^0(\mathbf{v}_{-i})$ is approximated by $\tilde{u}_i^{[k]}(\mathbf{v})$:

$$x_i(\tilde{\mathbf{v}}) = \frac{\tilde{u}_i^{[k]}(\tilde{\mathbf{v}}) - \frac{1}{r}(H_n^a(\tilde{V}) - H_n^a(\tilde{V} - \tilde{v}_i + v_i^0))}{R_n(P_n^a(\tilde{V}))} + \frac{\tilde{v}_i}{r}. \tag{28}$$

With this compensation, the utility (2) of owner i becomes (recalling the definition (21) of $\tilde{u}_i^{[k]}$)

$$\begin{aligned} u_i(\mathbf{x}(\cdot), \tilde{\mathbf{v}}, v_i) &= R_n(P_n^a(\tilde{V}))\left(x_i(\tilde{\mathbf{v}}) - \frac{v_i}{r}\right) \\ &= R_n(P_n^a(\tilde{V}))\left(\frac{\tilde{v}_i - v_i}{r}\right) + u_i^{[k]}(\tilde{\mathbf{v}}_{-i}) + \eta_i^{[k]}(\tilde{\mathbf{v}}) - \frac{H_n^a(\tilde{V}) - H_n^a(\tilde{V} + v_i^0 - \tilde{v}_i)}{r}. \end{aligned}$$

The first order condition of optimal report $\tilde{v}_i$ is then

$$\frac{\partial u_i(\tilde{\mathbf{v}}, v_i)}{\partial \tilde{v}_i} = R_n'(P^a(\tilde{V}))\frac{\partial P^a(\tilde{V})}{\partial V}\left(\frac{\tilde{v}_i - v_i}{r}\right) + \frac{\partial \eta_i^{[k]}(\tilde{\mathbf{v}})}{\partial \tilde{v}_i} = 0,$$

which results in the following optimal report:

$$\tilde{v}_i = v_i - \frac{r\frac{\partial \eta_i^{[k]}(\tilde{\mathbf{v}})}{\partial \tilde{v}_i}}{R_n'(P^a(\tilde{V}))\frac{\partial P^a(\tilde{V})}{\partial V}}.$$

We can also assess the deviation of the aggregate reported value from the true one,

$$\tilde{V} - V = -\frac{r\sum_i \frac{\partial \eta_i^{[k]}(\tilde{\mathbf{v}})}{\partial \tilde{v}_i}}{R_n'(P^a(\tilde{V}))\frac{\partial P^a(\tilde{V})}{\partial V}}. \tag{29}$$

Observe that $\sum_i \frac{\partial \eta_i^{[k]}(\tilde{\mathbf{v}})}{\partial \tilde{v}_i}$ is a weighted average of $\frac{\partial \epsilon_{i_1,\ldots,i_k}^{[k]}(\tilde{\mathbf{v}})}{\partial \tilde{v}_{i_1}}$ for various sequences $i_1, \ldots, i_k$. The latter can be approximated by (cf.(27)) $n^{-k+1}\left[\frac{1}{n}g_n^{(k+1)}(\bar{v})\prod_{l=1\ldots k}(v_{i_l} - m_{i_l}) + g_n^{(k)}(\bar{v})\right]$. With constant k and rising n, the first term in square brackets vanishes with n but the second does not, thus the numerator in (29) is $O(n^{-k+1})$. The denominator of (29) is asymptotically constant, thus we have $\tilde{V} - V = O(n^{-k+1})$. The expectation of $\tilde{V} - V$ over realizations of V has the same order of magnitude.

To assess the welfare loss associated with deviation from truthful reporting of types, we Taylor-expand the aggregate gains from trade $U_n^a(\tilde{V}, V) = R_n(P_n^a(\tilde{V}))\left(P_n^a(\tilde{V}) - \frac{V}{r}\right)$ around $U_n^a(V, V)$, noting that $\frac{\partial U_n^a(V,V)}{\partial \tilde{V}} = 0$, to obtain $U_n^a(\tilde{V}, V) - U_n^a(V, V) = \frac{1}{2}\frac{\partial^2 U_n^a(V,V)}{\partial \tilde{V}^2}(\tilde{V} - V)^2 + o(\tilde{V} - V)^2$. The aggregate gains from trade and all its derivatives are of order n, thus the aggregate welfare loss is $O(n^{-2k+3})$, so that per-owner welfare loss is $O(n^{-2k+2})$. For example, if the order of approximation is $k = 3$, doubling the number of owners reduces the per-owner welfare loss $2^6 = 64$ times.

4.2. The constitutional constraint

4.2.1. Problem statement

The discussion on p.19 has shown that the constitutional constraint (13) makes the first-best allocation impossible for large enough v_i, for those owners i to whom coercion is applied. We now characterize a mechanism which attempts to deliver the aggregate price as close as possible to the first-best, $P_n^a(V)$, while respecting the constitutional constraint (13), and assess its asymptotic properties. Section 4.1 has shown that achieving $P_n^a(V)$ exactly is generally impossible even when (13) is non-binding for all owners, because any mechanism attempting that would result in owners' deviation from truthful revelation of values. At the same time, such deviation can be made arbitrarily small as the number of owners increases. The current section assumes, for analytical tractability, that the owners always reveal their values truthfully.

Combine (17) with the first inequality in (19) to obtain

$$u_i^0(\mathbf{v}_{-i}) - \frac{H_n^a(V) - H_n^a(V + v_i^0 - v_i)}{r} + R_n(P_n^a(V))\frac{v_i - v_i^0}{r} \geq 0. \tag{30}$$

The derivative of the left-hand side of (30) with respect to v_i is equal to

$$R_n'(P_n^a(V))\frac{\partial P_n^a(V)}{\partial V}\frac{v_i - v_i^0}{r} \leq 0,$$

with strict inequality if the probability of transaction is positive. This means that if (30) is violated for some v_i, it is also violated for all values above v_i. Suppose that, whenever the first-best mechanism violates the constitutional constraint for owner i, the lowest possible compensation $\frac{v_i^0}{r}$ is given. Then, such compensation will be offered whenever the (reported) value v_i is equal or above the threshold $\hat{v}_i$ that turns (30) into an equality. Because the minimum compensation does not depend on v_i, we then have $\frac{\partial x_i(\mathbf{v})}{\partial v_i} = 0$ for all $v_i \geq \hat{v}_i$. At the same time, full differentiation of (17) with respect to v_i yields $R_n'(P(\mathbf{v}))\left(x_i(\mathbf{v}) - \frac{v_i}{r}\right)\frac{\partial P(\mathbf{v})}{\partial v_i} + R_n(P(\mathbf{v}))\frac{\partial x_i(\mathbf{v})}{\partial v_i} = 0$, which further means that $\frac{\partial P(\mathbf{v})}{\partial v_i} = 0$ for all $v_i \geq \hat{v}_i$. We then also have $\sum_{j\neq i}\frac{\partial x_j(\mathbf{v})}{\partial v_i} = \frac{\partial P(\mathbf{v})}{\partial v_i} - \frac{\partial x_i(\mathbf{v})}{\partial v_i} = 0$. While it is theoretically possible that a change in v_i leads to redistribution of compensations $x_j, j \neq i$, without changing the sum, it is not clear what such redistribution could achieve. We assume that $\frac{\partial x_j(\mathbf{v})}{\partial v_i} = 0, \forall j, \forall v_i \geq \hat{v}_i$. In other words, an owner i who reports a value above $\hat{v}_i$ is treated by the mechanism as if he has reported the value $\hat{v}_i$.

4.2.2. The mechanism and its properties

With these considerations, we can specify the optimal feasible mechanism as follows: deliver the first-best aggregate price $P_n^a(\cdot)$, baseline utility $u_i^0(\cdot)$ approximated by (cf.(21)) $\tilde{u}_i^{[k]}(\cdot)$, and compensations $x_i(\cdot)$ that satisfy (17), as if the owner i's valuation was

$$v_i^*(\mathbf{v}) = \min\{v_i, \hat{v}_i(\mathbf{v}_{-i})\},$$

rather than the true value v_i. Here $\hat{v}_i(\mathbf{v}_{-i})$ is defined by (cf.(30))

$$u_i^0(\mathbf{v}_{-i}^*) - \frac{H_n^a(\sum_{j\neq i} v_j^* + \hat{v}_i) - H_n^a(\sum_{j\neq i} v_j^* + v_i^0)}{r} + R_n(P_n^a(\sum_{j\neq i} v_j^* + \hat{v}_i))\frac{\hat{v}_i - v_i^0}{r} \equiv 0. \quad (31)$$

Such mechanism does not create incentives to deviate from truthful-telling, other than those stemming from imperfect approximation of $u_i^0(\mathbf{v}_{-i})$ and outlined in Section 4.1. We now provide some additional properties of the mechanism.

Proposition 5. *For any given realization of* $\mathbf{v}$*, coercion cannot be applied to all owners at the same time.*

Proof. Divide all owners into two groups: $i \in N_1$ have $v_i^* = v_i < \hat{v}_i$ and $i \in N_2$ have $v_i^* = \hat{v}_i \leq v_i$. In N_2, each owner i receives the minimum compensation $\frac{v_i^0}{r}$, thus the aggregate compensation to those in N_1 is $P_n^a(\sum_{\forall i} v_i^*) - \frac{\sum_{i\in N_2} v_i^0}{r}$. From the properties of $P_n^a(\cdot)$ we have $P_n^a(\sum_{\forall i} v_i^*) > \frac{\sum_{\forall i} v_i^*}{r} = \frac{\sum_{i\in N_1} v_i}{r} + \frac{\sum_{i\in N_2} \hat{v}_i}{r}$, so the aggregate surplus (compensation minus own value) of group N_1 is

$$P_n^a(\sum_{\forall i} v_i^*) - \frac{\sum_{i\in N_2} v_i^0}{r} - \frac{\sum_{i\in N_1} v_i}{r} \geq P_n^a(\sum_{\forall i} v_i^*) - \frac{\sum_{i\in N_2} \hat{v}_i}{r} - \frac{\sum_{i\in N_1} v_i}{r} > 0.$$

Therefore, N_1 is non-empty (otherwise it would have zero surplus), and at least some of its members have positive surplus. ■

Proposition 6. *An owner whose value is lowest possible,* v_i^0*, enjoys nonnegative gains from trade.*

Proof. Consider (30) with $v_i = v_i^0$ to conclude $u_i^0(\mathbf{v}_{-i}) \geq 0$. ■

How does x_i depend on v_i when $v_i \leq \hat{v}_i$, given $\mathbf{v}_{-i}$? Fully differentiating (17) with respect to v_i and rearranging, we obtain[4] $\frac{\partial x_i}{\partial v_i} = \frac{-R_n'(P_n^a)}{R_n(P_n^a)}\left(x_i - \frac{v_i}{r}\right)\frac{\partial P_n^a(V)}{\partial V}$. Because $\frac{\partial P_n^a(V)}{\partial V} > 0$ and $\frac{-R_n'(P_n^a)}{R_n(P_n^a)} > 0$, the sign of $\frac{\partial x_i}{\partial v_i}$ is equal to the sign of $x_i - \frac{v_i}{r}$, which is known to be decreasing

[4]The analysis is provided for the case when $v_j \leq \hat{v}_j$ for all j, thus $v_j^* = v_j$ is independent from v_i. The other scenario, when $v_j^* = \hat{v}_j$ for some j, is omitted.

from a positive to a negative value. In other words, x_i is increasing in v_i when i's gains from trade are positive, is flat when zero, and is decreasing when the gains from trade are negative.

What is the intuition behind this result? In non-coercive mechanisms (Myerson and Satterthwaite (1983) is a classical reference), compensation always increases with owner's value. This is because a higher requested compensation has two effects on welfare: more cash in case of a transaction (positive effect), but a lower probability that the transaction indeed happens (negative effect). For owners with higher value, the negative welfare effect is weaker while the positive one is the same, thus they ask more.

When coercion is applied to owners, the above logic is turned upside down. Coercion means that a lower probability of transaction has positive, not negative, effect on welfare. To preserve truthful revelation of types, the other effect should be reversed too: a higher reported type of an owner (which optimally entails a lower probability of transaction) should result in lower, not higher, compensation. Otherwise the owners being coerced would report infinitely high types.

4.2.3. Asymptotic properties

This section analyzes the asymptotic properties of the threshold $\hat{v}_i(\mathbf{v}_{-i})$, as the number of owners grows large, and welfare losses associated with deviation from the first-best strategy. From (24), the difference $H_n^a(\sum_{j\neq i} v_j^* + \hat{v}_i) - H_n^a(\sum_{j\neq i} v_j^* + v_i^0)$ in (31) can be approximated by $R_1\left(p^a\left(\frac{\sum_{j\neq i} v_j^* + \hat{v}_i}{n}\right)\right)(v_i - v_i^0) - \frac{1}{2n}R_1'\left(p^a\left(\frac{\sum_{j\neq i} v_j^* + \hat{v}_i}{n}\right)\right)(v_i - v_i^0)^2 + o(n^{-1})$, the first component of which is equal to $R_n(P_n^a(\sum_{j\neq i} v_j^* + \hat{v}_i))(v_i - v_i^0)$ in (31) and thus cancels out. Therefore, the threshold $\hat{v}_i(\mathbf{v}_{-i})$ satisfies

$$u_i^0(\mathbf{v}_{-i}^*) + \frac{1}{2nr}R_1'\left(p^a\left(\frac{\sum_{j\neq i} v_j^* + \hat{v}_i}{n}\right)\right)(\hat{v}_i - v_i^0)^2 + o(n^{-1}) = 0,$$

which allows us to approximate the threshold $\hat{v}_i$ as follows:

$$\hat{v}_i \approx \left(\frac{2nru_i^0(\mathbf{v}_{-i}^*)}{-R_1'\left(p^a\left(\frac{\sum_{j\neq i} v_j^* + \hat{v}_i}{n}\right)\right)} \right)^{\frac{1}{2}} + v_i^0. \tag{32}$$

Here $u_i^0(\mathbf{v}_{-i}^*)$ is approximated by $\tilde{u}_i^{[k]}(\{\hat{v}_i, \mathbf{v}_{-i}^*\})$, which depends on $\hat{v}_i$ via $\eta_i^{[k]}(\cdot)$, but Section 4.1.2 shows the latter is $O(n^{-k})$ and therefore has a vanishing impact on (32). $\tilde{u}_i^{[k]}(\cdot)$ also changes with n, but can be shown to converge to a constant. $p^a(\cdot)$ in (32) is asymptotically constant and independent of $\hat{v}_i$, too. Therefore, we have that $\hat{v}_i = O(n^{\frac{1}{2}})$.

The welfare loss associated with departure from the first-best mechanism is due to the difference between the true average value and the "truncated" one,

$$\frac{\sum_{i=1}^n v_i^*(\mathbf{v}) - v_i}{n} = -\frac{\sum_{i=1}^n I(v_i \geq \hat{v}_i(\mathbf{v}_{-i}))(v_i - \hat{v}_i(\mathbf{v}_{-i}))}{n}. \tag{33}$$

Asymptotically, such difference depends on the (generally unknown) distributions of owner valuations: the thicker are the tails of the distributions, the more likely that $v_i^* \neq v_i$, the further away from zero is (33). Specifically, for distributions without tails (i.e. bounded from above), $\hat{v}_i$ eventually becomes greater than the distribution upper bound for all i, and the constitutional constraint does not reduce welfare for large enough n.

For distributions that have tails, if all elements of the sum in (33) converge to zero at the same rate, then so does (33) which is their average. Suppose the owners' distribution is Pareto, known for its heavy tails, such that $\Pr(v_i \geq x) = \left(\frac{v_i^0}{x}\right)^\alpha$. Then the expectation of $v_i^*(\mathbf{v}) - v_i$ in (33), conditional on $\mathbf{v}_{-i}$, is given by $\frac{\alpha}{\alpha-1}(v_i^0)^\alpha \hat{v}_i(\mathbf{v}_{-i})^{-(\alpha-1)}$, which is $O(n^{-\frac{\alpha-1}{2}})$. The discrepancy between the buyer's price per owner $p^a\left(\frac{\sum_1^n v_i^*}{n}\right)$ and the optimal one $p^a(\bar{v})$ is also $O(n^{-\frac{\alpha-1}{2}})$. The discrepancy between actual and maximal owners' gains from trade is $O(n^{-(\alpha-1)})$.

5. Numerical examples

We now illustrate the proposed mechanism with two numerical examples. In both examples, owner values were randomly drawn from Pareto distributions such that $\Pr(v_i \geq x) = \left(\frac{2m_i}{2m_i+x}\right)^3, \forall x \geq 0$, with means equal to m_i. In Example 1, there are four "type-A" owners with $m_i = 1$ and two "type-B" owners with $m_i = 2$. In Example 2, there are 40 type-A owners and 20 type-B owners. For comparison purposes, the first four type-A owners and two type-B owners in Example 2 have the same values as those in Example 1.

We assume that the prospective buyers show up, on average, once per year. The discount rate is 5%. Thus, $r = 0.05$. We approximate u_i^0 by $\tilde{u}_i^{[k]}$ with $k = 3$.

The buyers' distribution of values is uniform between zero and some P_m such that the mean buyer's value $\frac{P_m}{2}$ is equal to the aggregate mean of all owners NPVs, $\frac{\sum_i m_i}{r}$, in both examples. Specifically, $P_m = 320$ in Example 1 and $P_m = 3200$ in Example 2.

For comparison purposes, we also illustrate the fixed compensation mechanism.

Results for individual owners are reported in Table 1. In Example 2, we show only owners whose values coincide with those from Example 1, as well as those with minimal and maximal values within each type. In all examples, the constitutional constraint was not binding for all owners. Average outcomes are given in Table 2. Figure 1 illustrates how the compensation x_{A_1} and the aggregate price P respond to variation of the reported type $\tilde{v}_{A_1}$ by owner A_1.

All above tables and figures confirm the findings of the paper. The compensation x_i first rises and then declines with the reported type $\tilde{v}_i$, but x_i varies less with $\tilde{v}_i$ as the number of owners grows. At the same time, the aggregate price P rises with $\tilde{v}_i$, with the dependence becoming increasingly linear. In both optimal and fixed-compensation mechanisms, the deviation of all relevant parameters from the first-best decreases with the number of owners, but in the optimal mechanism this deviation is always smaller and converges to zero at a much faster rate.

Owner	v_i	Report bias, $\tilde{v}_i - v_i$		Compensation, x_i			Gains from trade, u_i		
		Ex1	Ex2	Ex1	Ex2	F	Ex1	Ex2	F
A_1	0.7439	0.1511	0.0001	40.050	35.972	35.367	14.67	14.22	14.31
A_2	1.8319	0.1223	0.0001	39.371	35.960	35.367	1.59	-0.46	-0.89
A_3	4.8276	0.0917	0.0001	30.618	35.537	35.367	-38.43	-41.13	-42.74
A_4	0.6683	0.1539	0.0001	40.026	35.970	35.367	15.54	15.24	15.37
$A_{\min}$	0.0094		0.0001		35.936	35.367		24.10	24.57
$A_{\max}$	5.1759		0.0001		35.452	35.367		-45.89	-47.60
B_1	3.0314	0.2241	0.0002	77.881	71.872	70.734	10.06	7.58	7.06
B_2	0.4510	-0.096	0.0001	69.693	71.779	70.734	35.37	42.31	43.11
$B_{\min}$	0.0146		0.0001		71.718	70.734		48.15	49.20
$B_{\max}$	11.213		0.0001		69.353	70.734		-104.43	-107.24

Table 1: Mechanism outcomes for individual owners. Ex(i) is optimal mechanism for Example i, F is fixed compensation mechanism.

	Example 1		Example 2	
	OPT	F	OPT	F
Average value, $\bar{v}$	1.9257		1.5443	
1st best average price, $p^a(\bar{v})$	49.1714		47.8165	
1st best average gains from trade	6.4958		11.4133	
Average report bias, $\frac{\tilde{V}-V}{n}$	0.1079	N/A	0.0001	N/A
Average price, relative to 1st best	0.4352	-2.0157	0.0003	-0.6608
Avg gains from trade, relative to 1st best	-0.0296	-0.4594	-1.4×10^{-8}	-0.0494

Table 2: Average mechanism outcomes. OPT is optimal, F is fixed compensation mechanism.

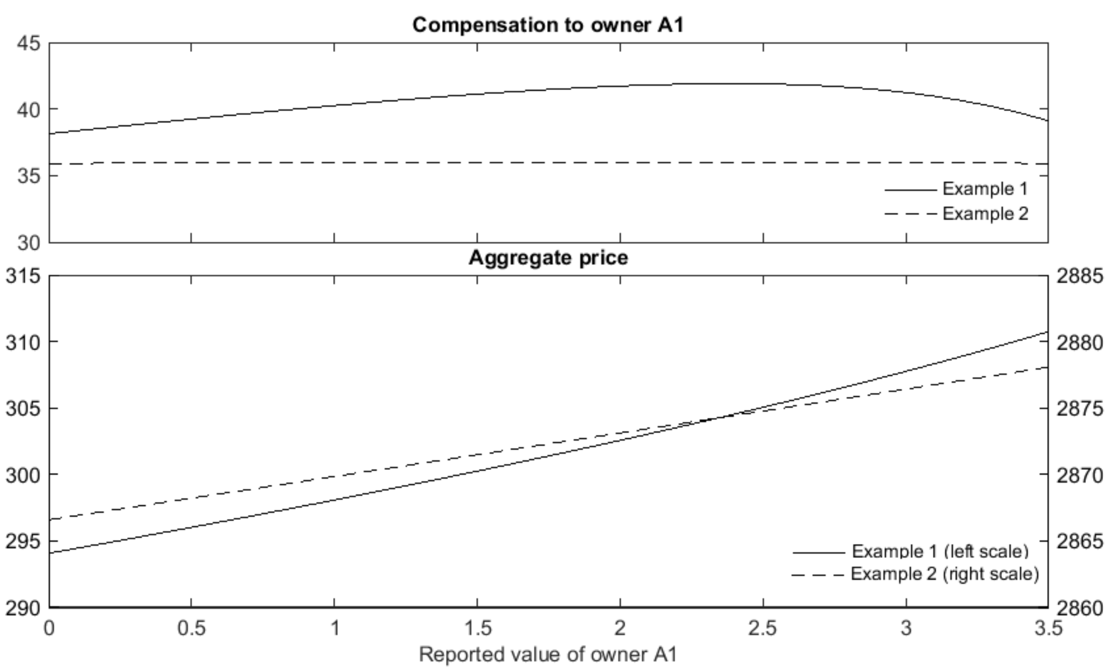

Figure 1: Effects of varying reported value $\tilde{v}_{A_1}$ by owner A_1.

6. Discussion

6.1. Optimality of the mechanism

Although we refer to the mechanism developed in this paper as "optimal," the paper offers no proof that it maximizes the ex-ante gains from trade. This is because calculation of such ex-ante gains from trade requires knowledge of owners' value distributions, which we avoid using throughout the paper. If some assumptions about owners' distributions were employed, a (slightly) better mechanism is theoretically possible. However, it may be more computationally intensive, its asymptotic properties are likely to be the same, and it may underperform if owners' distributions are misspecified.

6.2. Misspecification

How does the mechanism perform if the government has incorrectly estimated the owners' mean values m_i or the buyers' distribution of values F_n? For comparison purposes, first observe that, in each of these misspecification cases, the fixed-compensation mechanism results in a biased aggregate price (15). The bias generally does not go away with rising number of owners.

In the optimal mechanism, owner means m_i affect only the approximation of $u_i^0(\mathbf{v}_{-i})$. The effect is twofold: through the shares of each owner $s_{i,n}$, and through the approximation error $\epsilon^{[k]}_{i_1,\ldots,i_k}(\mathbf{v})$. If all owner means are misspecified by the same factor, such that the estimated mean $\hat{m}_i$ equals qm_i for some $q > 0$, then the shares $s_{i,n}$ are unchanged, and the only effect is through $\epsilon^{[k]}_{i_1,\ldots,i_k}(\mathbf{v})$. The latter is approximated by (27); replacing the true m_i in the latter by its estimate $\hat{m}_i$ yields $\epsilon^{[k]}_{i_1,\ldots,i_k}(\mathbf{v}) \approx n^{-k+1} g_n^{(k)}(\bar{v}) \prod_{l=1\ldots k}(v_{i_l} - \hat{m}_{i_l})$, which is biased from zero by $n^{-k+1} g_n^{(k)}(E\bar{v}) \prod_{l=1\ldots k}(\hat{m}_{i_l} - m_{i_l})$. The bias converges to zero at the same rate as the error itself does, and therefore, although the reported value deviation (29) now has generally non-zero expectation, it converges to zero at the same rate as before. We can conclude that misspecified owners' average values do not compromise asymptotic properties of the optimal mechanism.

What if the mechanism designer's estimate of buyers distribution of values $\hat{F}(\cdot)$ (dropping the subscript n) is incorrect? Denote by $\hat{R}(P) \equiv \frac{1-\hat{F}(P)}{r+1-\hat{F}(P)}$ the estimate of $R(P)$. Then, if all owners' reported values $\tilde{v}_i$ were true, the estimate $\hat{P}^a(\tilde{V})$ of the first-best price $P^a(V)$ that satisfies

$$\hat{R}'(\hat{P}^a)\left(\hat{P}^a - \frac{\tilde{V}}{r}\right) + \hat{R}(\hat{P}^a) = 0 \tag{34}$$

would generally deviate from true first-best (4). However, if the owners know the true distribution $F(\cdot)$, a bias in the mechanism will generally cause them to deviate from reporting true types. Denote by $\hat{H}^a(\tilde{V}) = \int_0^{\tilde{V}} \hat{R}(\hat{P}^a(z))\mathrm{d}z$ the mechanism designer's estimate of $H^a(\tilde{V})$. Ignoring the errors of approximation of $u_i^0(\mathbf{v}_{-i})$, and assuming the constitutional constraint (13) does not bind any owner, the compensation to owner i is calculated as (cf.(28))

$$x_i(\tilde{\mathbf{v}}) = \frac{u_i^0(\tilde{\mathbf{v}}_{-i}) - \frac{1}{r}(\hat{H}^a(\tilde{V}) - \hat{H}^a(\tilde{V} - \tilde{v}_i + v_i^0))}{\hat{R}(\hat{P}^a(\tilde{V}))} + \frac{\tilde{v}_i}{r}.$$

Such owner maximizes $R(\hat{P}^a(\tilde{V}))\left(x_i(\tilde{\mathbf{v}}) - \frac{v_i}{r}\right)$, which results in the following first-order condition of optimal $\tilde{v}_i$:

$$\left(\frac{R'(\hat{P}^a(\tilde{V}))}{R(\hat{P}^a(\tilde{V}))} - \frac{\hat{R}'(\hat{P}^a(\tilde{V}))}{\hat{R}(\hat{P}^a(\tilde{V}))}\right)\frac{R(\hat{P}^a(\tilde{V}))}{\hat{R}(\hat{P}^a(\tilde{V}))}\left[u_i^0(\mathbf{v}_{-i}) - \frac{1}{r}(\hat{H}^a(\tilde{V}) - \hat{H}^a(\tilde{V} - \tilde{v}_i + v_i^0))\right] \\ + \frac{R'(\hat{P}^a(\tilde{V}))}{r}(\tilde{v}_i - v_i) = 0.$$

Aggregating the above across all owners, and recalling that the term in square brackets aggregated across all i equals $\hat{R}(\hat{P}^a(\tilde{V}))\left(\hat{P}^a(\tilde{V}) - \frac{\tilde{V}}{r}\right)$, we obtain

$$\left(\frac{R'(\hat{P}^a(\tilde{V}))}{R(\hat{P}^a(\tilde{V}))} - \frac{\hat{R}'(\hat{P}^a(\tilde{V}))}{\hat{R}(\hat{P}^a(\tilde{V}))}\right)R(\hat{P}^a(\tilde{V}))\left(\hat{P}^a(\tilde{V}) - \frac{\tilde{V}}{r}\right) + \frac{R'(\hat{P}^a(\tilde{V}))}{r}(\tilde{V} - V) = 0.$$

The latter expression can be modified to

$$R'(\hat{P}^a(\tilde{V}))\left(\hat{P}^a(\tilde{V})-\frac{V}{r}\right)-\frac{\hat{R}'(\hat{P}^a(\tilde{V}))}{\hat{R}(\hat{P}^a(\tilde{V}))}R(\hat{P}^a(\tilde{V}))\left(\hat{P}^a(\tilde{V})-\frac{\tilde{V}}{r}\right)=0.$$

Because $\hat{P}^a(\tilde{V})$ is chosen to achieve (34), we can also conclude $R'(\hat{P}^a(\tilde{V}))\left(\hat{P}^a(\tilde{V})-\frac{V}{r}\right)+R(\hat{P}^a(\tilde{V}))=0$, which means (cf.(4)) $\hat{P}^a(\tilde{V})=P^a(V)$. In other words, the deviation of the owners' report from the truth is such that first-best optimality of aggregate compensation is preserved.

We can conclude that the optimal mechanism proposed in this paper is robust to mistakes by the government in estimation of the mechanism's inputs.

6.3. Collusion

Does the optimal mechanism encourage collusion? Because the ultimate purpose of the mechanism is to maximize the owners' joint gains from trade, collusion between owners is unlikely. Does a buyer have an incentive to collude with a subset of owners, for example by secretly purchasing their units and manipulating their prices? Because manipulation with reported type $\tilde{v}_i$ generally affects payments to owners other than i, a buyer who got control of unit i might have an incentive to engage in such manipulation. However, if a buyer attempts to manipulate the reported values of a sizeable fraction of all units, then the remaining independent owners would observe an unusual distribution of reported types. From that, they would infer that somebody is certain to assemble their property, which would lead them to reevaluate the probability of transaction and adjust their reported types so that the buyer's price is increased. This would offset the potential gains from manipulation. Further research is required to assess the potential magnitude of manipulation and its welfare consequences.

6.4. Auctions

When the proposed mechanism is initiated for the first time for a given set of units to be assembled, multiple buyers may show up simultaneously. In this case, an auction can be administered, with the reservation price being equal to the one determined by the optimal mechanism. Because the final transaction price is then determined by competition between buyers, rather than by inputs of owners, the difference between the final and the reservation prices can be divided between owners in an arbitrary predetermined way, for example proportionately to their mean values m_i.

6.5. Fixed buyer's price

A scenario opposite to the previous section is when there is only one buyer with non-random commonly known purchase price. In particular, this scenario is relevant when the purchase is initiated by a government with transparent budget. The mechanism of this paper depends heavily on the random nature of buyers' valuations, and cannot be used in this scenario. Further research is required to address the issue.

Appendix A. Proofs

Proof of Proposition 1. Because the proof is provided for a given number of owners, we drop the subscript n throughout the proof for clarity of exposition. First, observe that Assumption 1 implies

$$F^{(2)}(P)(1-F(P))+2(F'(P))^2>0, \forall P. \tag{A.1}$$

We have the following relationships between $R(P)$ and its derivatives:

$$\begin{aligned} R'(P) &= -\frac{rF'(P)}{(r+1-F(P))^2}, \\ R^{(2)}(P) &= -\frac{r}{(r+1-F(P))^2}\left(F^{(2)}(P)+\frac{2(F'(P))^2}{r+1-F(P)}\right)\underbrace{<}_{(A.1)}\frac{2(R'(P))^2}{R(P)}. \end{aligned} \tag{A.2}$$

Calculate the second derivative of (3) with respect to P at every P^a that satisfies (4) as follows:

$$R^{(2)}(P^a)\left(P^a - \frac{V}{r}\right) + 2R'(P^a) \underbrace{<}_{(A.2)} 2R'(P^a)\left(\frac{R'(P^a)}{R(P^a)}\left(P^a - \frac{V}{r}\right) + 1\right) \underbrace{=}_{(4)} 0$$

Therefore, at any point P^a where the first derivative of $U(P, \mathbf{v}, V)$ with respect to P is zero, the second derivative is negative. Because $U(P, \mathbf{v}, V)$ is a continuous function of its arguments, such point P^a is unique for a given $\mathbf{v}$ and corresponds to the global maximum.■

Proof of Proposition 3. Suppose the contrary, that for some $v_i, \mathbf{x}_{-i}$, we have $x_i(v_i, \mathbf{x}_{-i}) > x_i^*(v_i, \mathbf{x}_{-i})$. Consider changing $x_i(v_i, \mathbf{x}_{-i})$ by a marginal amount $-\mathrm{d}x < 0$. Because the compensation becomes closer to $x_i^*(v_i, \mathbf{x}_{-i})$, owner i is better off.

If all other owners enjoy positive gains from trade, they are better off, too, because lowered x_i increases the probability of transaction. Thus, we have achieved a Pareto-improvement, compromising the initial assumption of optimality.

Suppose there exists an owner j with value v_j who enjoys negative gains from trade: $x_j(v_j, \mathbf{x}_{-j}) < \frac{v_j}{r}$. Because $x_j^*(v_j, \mathbf{x}_{-j}) > \frac{v_j}{r}$, we also have that $x_j(v_j, \mathbf{x}_{-j}) < x_j^*(v_j, \mathbf{x}_{-j})$. Consider changing $x_j(v_j, \mathbf{x}_{-j})$ by a marginal amount $\mathrm{d}x > 0$. Owner j is better off, as the compensation becomes closer to his ideal $x_j^*(v_j, \mathbf{x}_{-j})$. Because x_i and x_j have changed by the same amount but in opposite directions, the total transaction price is unaffected, so the welfare of owners other than i, j is unaffected, too. We have achieved a Pareto-improvement, again, thus the initial allocation could not be optimal.■

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